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MATHEMATICS
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PAPER 3 – SERIES AND DIFFERENTIAL EQUATIONS

Monday 9 May 2011 (morning)

1 hour

INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- Answer all the questions.
- Unless otherwise stated in the question, all numerical answers must be given exactly or correct to three significant figures.

Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working, e.g. if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. *[Maximum mark: 10]*

(a) Find the first three terms of the Maclaurin series for $\ln(1+e^x)$. *[6 marks]*

(b) Hence, or otherwise, determine the value of $\lim_{x\to 0}\frac{2\ln(1+e^x)-x-\ln 4}{x^2}$. *[4 marks]*

2. *[Maximum mark: 8]*

Consider the differential equation $\frac{dy}{dx}=x^2+y^2$ where $y=1$ when $x=0$.

(a) Use Euler's method with step length 0.1 to find an approximate value of y when $x=0.4$. *[7 marks]*

(b) Write down, giving a reason, whether your approximate value for y is greater than or less than the actual value of y. *[1 mark]*

3. *[Maximum mark: 11]*

Solve the differential equation

$$x^2\frac{dy}{dx}=y^2+3xy+2x^2$$

given that $y=-1$ when $x=1$. Give your answer in the form $y=f(x)$.

4. *[Maximum mark: 15]*

The integral I_n is defined by $I_n = \int_{n\pi}^{(n+1)\pi} \mathrm{e}^{-x} \,|\sin x|\, \mathrm{d}x$, for $n \in \mathbb{N}$.

(a) Show that $I_0 = \frac{1}{2}(1 + \mathrm{e}^{-\pi})$. *[6 marks]*

(b) By letting $y = x - n\pi$, show that $I_n = \mathrm{e}^{-n\pi} I_0$. *[4 marks]*

(c) Hence determine the exact value of $\int_0^{\infty} \mathrm{e}^{-x} \,|\sin x|\, \mathrm{d}x$. *[5 marks]*

5. *[Maximum mark: 16]*

The exponential series is given by $\mathrm{e}^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

(a) Find the set of values of x for which the series is convergent. *[4 marks]*

(b) (i) Show, by comparison with an appropriate geometric series, that

$$\mathrm{e}^x - 1 < \frac{2x}{2-x}, \text{ for } 0 < x < 2.$$

(ii) Hence show that $\mathrm{e} < \left(\frac{2n+1}{2n-1}\right)^n$, for $n \in \mathbb{Z}^+$. *[6 marks]*

(c) (i) Write down the first three terms of the Maclaurin series for $1 - \mathrm{e}^{-x}$ and explain why you are able to state that

$$1 - \mathrm{e}^{-x} > x - \frac{x^2}{2}, \text{ for } 0 < x < 2.$$

(ii) Deduce that $\mathrm{e} > \left(\frac{2n^2}{2n^2 - 2n + 1}\right)^n$, for $n \in \mathbb{Z}^+$. *[4 marks]*

(d) Letting $n = 1000$, use the results in parts (b) and (c) to calculate the value of e correct to as many decimal places as possible. *[2 marks]*